

# Spinors and Octonions

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## Abstract

Octonions are introduced through some spin representations. The groups  $G_2$ ,  $F_4$  and the  $E$  series appear in a natural manner; one way to understand octonions is as the “second coming” of the reals, but with the spinors instead of vectors. Some physical applications in  $M$ - and  $F$ -theory as putative “theories of everything” are suggested.

## 1 The Seven Sphere

Consider the real, complex and quaternion numbers  $R$ ,  $C$ ,  $H$ . Identify the normed vector spaces  $R^{8k} \cong C^{4k} \cong H^{2k}$ , and write the natural inclusion of the isometry groups

$$O(8k) \supset U(4k) \supset Sp(2k) \quad (1)$$

Recall now the *Spin groups*, which cover the rotation groups twice,  $Spin(n)/Z_2 = SO(n)$ ; remind only that  $Spin(7)$  has a real 8-dim irreducible representation; as  $SO(n)$  is the maximal isometry group for spheres, the previous sequence becomes for  $k = 1$

$$\begin{array}{ccccccc} Sp(2) & \subset & SU(4) & \subset & Spin(7) & \subset & SO(8) \\ H & & C & & (O) & & R \end{array} \quad (2)$$

where the adscription of division algebras other than  $(O)$  is clear. It is fairly easy to see that all these group act *trans* on the seven sphere of constant norm vectors in  $R^8$ ; therefore after finding the stabilizers we get

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\*To Alberto Galindo in his seventieth birthday

$$SO(8)/SO(7) = Spin(7)/G_2 = SU(4)/SU(3) = Sp(2)/Sp(1) = S^7 \quad (3)$$

where  $G_2$  is *defined* to be the little group of  $Spin(7)$  acting on  $S^7$ ; the other isotropies are obvious. From the above inclusions (2) we get *four* commutative diagrams

O vs. R

$$\begin{array}{ccccc} G_2 & \rightarrow & Spin(7) & \rightarrow & S^7 \\ \downarrow & & \downarrow & & \parallel \\ SO(7) & \rightarrow & SO(8) & \rightarrow & S^7 \\ \downarrow & & \downarrow & & \\ RP^7 & = & RP^7 & & \end{array} \quad (4)$$

where we learn from the first column an interesting result.

H vs. C

$$\begin{array}{ccccc} Sp(1) & \rightarrow & Sp(2) & \rightarrow & S^7 \\ \downarrow & & \downarrow & & \parallel \\ SU(3) & \rightarrow & SU(4) & \rightarrow & S^7 \\ \downarrow & & \downarrow & & \\ S^5 & = & S^5 & & \end{array} \quad (5)$$

where the vertical lines are obvious:  $SU(3)/SU(2) = S^5 = Spin(6)/Spin(5)$ : we learn *passim* the Cartan identities

$$Sp(1) = Spin(3) = SU(2), \quad Spin(5) = Sp(2) \quad \text{and} \quad Spin(6) = SU(4) \quad (6)$$

C vs. O

$$\begin{array}{ccccc} SU(3) & \rightarrow & SU(4) & \rightarrow & S^7 \\ \downarrow & & \downarrow & & \parallel \\ G_2 & \rightarrow & Spin(7) & \rightarrow & S^7 \\ \downarrow & & \downarrow & & \\ S^6 & = & S^6 & & \end{array} \quad (7)$$

and we learn of the 7-dim irrep of  $G_2$ , as  $S^6 \subset R^7$ . From (3) we already know  $\dim G_2 = 14$ . Now the dimension of 3-forms in  $R^7$  are  $\binom{7}{3} = 35 = 7^2 - 14$ : hence  $G_2$  leaves invariant a generic three-form in  $R^7$  (this becomes the octonion multiplication, of course, and with a little extra effort we conclude that  $G_2$  is the automorphism group of a multiplicative structure: because of the metric, we trade the 3-form  $T_3^0$  by a  $T_2^1$  tensor, which characterizes algebras; this is our way of introducing octonions!).

There is a fourth diagram

C vs. R

$$\begin{array}{ccccc}
SU(3) & \rightarrow & SU(4) & \rightarrow & S^7 \\
\downarrow & & \downarrow & & \parallel \\
SO(7) & \rightarrow & SO(8) & \rightarrow & S^7 \\
\downarrow & & \downarrow & & \\
M_{13} & = & M_{13} & & 
\end{array} \tag{8}$$

where  $M_{13}$  is a symmetric space. We shall see that there is sense in calling  $Spin(7) \sim Oct(1)$ .

The relation (1) holds for any  $k$ . For  $k = 2$  it is expanded to

$$\begin{array}{ccccccc}
Sp(4) & \subset & SU(8) & \subset & Spin(9) & \subset & SO(16) \\
H & & C & & (O) & & R
\end{array} \tag{9}$$

as, again,  $\dim Spin(9) = 16$ . We shall only reproduce the  $Spin(9) = "Oct(2)" \subset O(16)$  relation for  $k = 2$

O vs. R

$$\begin{array}{ccccc}
Spin(7) & \rightarrow & Spin(9) & \rightarrow & S^{15} \\
\downarrow & & \downarrow & & \parallel \\
SO(15) & \rightarrow & SO(16) & \rightarrow & S^{15} \\
\downarrow & & \downarrow & & \\
M_{84} & = & M_{84} & & 
\end{array} \tag{10}$$

We do not know of any interpretation of the  $M_{84}$  manifold, except that reminds one of the 3-forms in 11 dimensions (9 effective) associated to the membranes in  $M$  theory. We shall also see that there is a sense in calling  $Spin(9) \sim Oct(2)$ .

For a good reference on Spinor groups look at Porteous's book [1].

## 2 Octonions

Granted that the multiplicative structure we found before is invertible, but nonassociative, we have the division algebra of octonions ( $O$ ). We know also

i) The automorphism group of ( $O$ ) is  $G_2$ . There is a natural 7-dim representation acting upon the imaginary octonions (of course, the real part is elementwise invariant); also,  $G_2$  respects the octonion norm, hence the 6-sphere remains invariant: this fact leads up to endowing the 6-sphere with a (quasi!)-complex structure (no complex, because no symplectic: the spheres are 2-connected!). The construction of quaternions can be carried out exactly equal: in  $R^3$  live the imaginary quaternions, and imaginary quaternion product

(vector product) becomes a 3-form, hence a volume form. The invariance group is  $SL(3, R)$ , but there is also an invariant norm, hence

$$Aut H = SL(3, R) \cap O(3) = SO(3) \quad (11)$$

ii) If the octonion division algebra were associative,  $S^7$  will be  $Oct(1)$ , in the sense of the previous true relations

$$O(1) = S^0, \quad U(1) = S^1, \quad Sp(1) = S^3, \quad (12)$$

But here  $Oct(1) = S^7$  gets stabilized through the automorphism group  $G_2$  to become the *twisted* sphere structure  $Spin(7) \cong S^3(\times S^7(\times S^{11}$  or

$$Oct(1) = Spin(7) \quad (13)$$

I explained these twists in [2]. Let us look for the projective line  $OP^1$ . For the other algebras we have

$$\begin{aligned} RP^1 &= S^1 = O(2)/O(1)^2, \quad CP^1 = S^2 = U(2)/U(1)^2, \\ HP^1 &= S^4 = Sp(2)/Sp(1)^2 \end{aligned} \quad (14)$$

But here we get, instead

$$OP^1 = S^8 = Spin(9)/Spin(8) = Oct(2)/Oct(1)^2 \quad (15)$$

Does it make sense to call  $Oct(1)^2 \sim Spin(8)$ ? Yes:  $Oct(1)^2$  should be  $S^7 \times S^7$  stabilized by  $G_2$  again; and indeed in twisted spheres

$$Spin(8) = S^3(\times S^7(\times S^7(\times S^{11}, \quad Spin(9) = S^3(\times S^7(\times S^{11}(\times S^{15} \quad (16)$$

as sphere structure, so also  $Spin(9) \sim Oct(2)$ .

iii) To complete the issue, we reach  $Oct(3)$  but not more, because nonassociativity prohibits this (the mathematical reason is told below). It turn out that

$$Oct(3) = F^4 \cong S^3(\times S^{11}(\times S^{15}(\times S^{23} \quad (17)$$

and

$$OP^2 = F_4/B_4 \cong S \cdot Oct(3)/Oct(2) = F_4/Spin(9) \quad (18)$$

So the factor of  $Oct(1)$  dissapears as  $O(1)$  disssapears in

$$RP^n = O(n+1)/O(n) \times O(1) = SO(n+1)/O(n) \quad (19)$$

All this reinforces the octonions as : 1) The second coming of the reals, after dimension 8; and also 2) With spinors, not with the vectors of the orthogonal group.

For general introduction of octonions for physicist see [3] and [4]; some of the identifications above are in [5].

### 3 The Magic Square

The natural inclusions, for any  $n$

$$\begin{array}{ccccccc} O(n) & \subset & U(n) & \subset & Sp(n) & & \\ & & & & \cap & & \\ & & & & U(2n) & & \\ & & & & \cap & & \\ & & & & O(4n) & & \end{array} \quad (20)$$

are the key to understand symmetric spaces, as we have shown elsewhere [5]. There are seven classes of (*classical*) symmetric spaces, the four associated to the previous diagram, namely

$$U(n)/O(n), \quad Sp(n)/U(n), \quad U(2n)/Sp(n) \quad \text{and} \quad O(2n)/U(n) \quad (21)$$

and the three families associated to different “floors”  $n = p + q$ :

$$K(p+q)/K(p) \times K(q), \quad \text{with } K = R, C \text{ or } H, \quad (22)$$

which are all grassmannians (projective spaces for  $q = 1$ ). Cartan found all this around 1926, and he even went further and classified the exceptional symmetric spaces; the standard source is [6].

To understand these, recall the octonions, although nonassociative, are alternative: the associator of three octonions

$$[a, b, c] := a(bc) - (ab)c \quad (23)$$

is fully antisymmetric. Therefore the symmetric algebra is a (conmutative) Jordan algebra. *The exceptional groups (except  $G_2$ ) are automorphism groups of certain Jordan algebras over the octonions.*

We form first the mutilated square for the first floor in (20)

$$\begin{array}{ccccccc}
O(1) & \subset & U(1) & \subset & Sp(1) & \subset & Oct(1) = Spin(7) \\
\cap & & \cap & & \cap & & \\
U(1) & \subset & U(1)^2 & \subset & U(2) & & \\
\cap & & \cap & & \cap & & \\
Sp(1) & \subset & U(2) & \subset & O(4) & & 
\end{array} \tag{24}$$

extending it in a natural way. For  $n = 2$ , it is completed:

$$\begin{array}{ccccccc}
O(2) & \subset & U(2) & \subset & Sp(2) & \subset & Oct(2) = Spin(9) \\
\cap & & \cap & & \cap & & \cap \\
U(2) & \subset & U(2)^2 & \subset & U(4) & \subset & Spin(10) \\
\cap & & \cap & & \cap & & \cap \\
Sp(2) & \subset & U(4) & \subset & O(8) & \subset & Spin(12) \\
\cap & & \cap & & \cap & & \cap \\
Spin(9) & \subset & Spin(10) & \subset & Spin(12) & \subset & Spin(16)
\end{array} \tag{25}$$

The last column can be understood as complexification, quaternionization and octonionization of  $Spin(9) = Oct(2)$ ; see Freudenthal [7] or [4].

And now the exceptional groups are obtained from the last column/row: as we generate  $O(n+1)$  by adding the vector irrep to the adjoint irrep (check  $\binom{n+1}{2} = \binom{n}{2} + n$ ), we trade vector by spinor:

$F_4$  is the extension of  $Spin(9) = Oct(2)$ : adjoint + spin,  $36 + 16 = 52$

As we get  $SU(n+1)$  from  $SU(n)$  by adding the vector and a  $U(1)$  factor to the adjoint  $((n+1)^2 - 1 = n^2 - 1 + 2n + 1)$ , here we get  $E_6$ :

$E_6$  is the extension of  $O(10)$  with the  $Spin + U(1)$ :  $45 + 1 + 32 = 78$

As we get  $Sp(n+1)$  from  $Sp(n)$  by adding the vector and a  $Sp(1)$  factor, here we get  $E_7$ :

$E_7$  extends  $O(12)$ :  $adj + Sp(1) + spin$ ,  $66 + 3 + 64 = 133$

Finally, the octonionic extension requires only the spin irrep, as octonions behave like the reals:

$E_8$  extends  $O(16)$ :  $adj + spin$ ;  $120 + 128 = 248$

One checks the skew square of the spin irrep contains the adjoint, and that the Jacobi identity is satisfied ( a nontrivial task!).

This allows us to write the *Freudenthal magic square* in the conventional form

$$\begin{array}{cccc}
O(3) & U(3) & Sp(3) & F_4 \\
U(3) & U(3)^2 & U(6) & E_6 \\
Sp(3) & U(6) & O(16) & E_7 \\
F_4 & E_6 & E_7 & E_8
\end{array} \tag{26}$$

So we see that the exceptional groups (except  $G_2$ ) are extensions of some orthogonal groups by the spin representation: the character  $O$ ,  $U$ ,  $Sp$  or  $Oct$  is reflected in the 0,  $U(1)$ ,  $Sp(1)$ , 0 added factors. For  $G_2$ , it is the little group of  $Spin(7)$  acting in the seven sphere, as said. This relation between exceptional groups and spinors, which I learned from [8], remains a bit mysterious.

The symmetric spaces involving the exceptional groups are now very clear; there are twelve of them. We shall only exhibit the four associated with  $E_6$ . They are the quotients in the graphs (25) and (26)

$$\begin{array}{ccc}
& F_4 & Spin(10) \\
& \cap & \cap \\
SU(6) \subset E_6 & = & E_6
\end{array} \tag{27}$$

and account for three of them, the fourth is associated to the split form, and it is  $E_6/Sp(4)$ . We shall consider one of them:

$$E_6/O(10) \cdot U(1) \tag{28}$$

that is, the complexification of the Moufang plane, with 32 dimensions. It is the simplest (rank two) hermitian exceptional symmetric space.

## 4 Supersymmetry and F-Theory

After this excursion in pure dry mathematics, it is healthy to inject some physics. The idea is that projective geometry could play a role in the real world!

The Moufang octonionic plane considered before

$$OP^2 = F_4/B_4 = F_4/Spin(9) \tag{29}$$

ends the series  $RP^2 = SO(3)/O(2)$ ,  $CP^2 = SU(3)/U(2)$ , and  $HP^2 = Sp(3)/Sp(1) \cdot Sp(2)$ . Projective spaces of dimension higher than two are necessarily desargian, which implies the underlying number field is associative; hence, the octonion groups stop at three. Now for some physics, in concrete 11-dim supergravity. P. Ramond [9] has shown that the maximal supergravity multiplet (triplet) in 11 dimensions (9 transverse)

$$\begin{array}{ccc}
graviton h & -gravitino \psi & +3 - form C \\
44 & -128 & +84
\end{array} \tag{30}$$

corresponds to the three embeddings of  $B_4$  in  $F_4$  : this “3” is precisely the Euler number of  $OP^2$  (with homology only in zero, eight and 16 dimensions,  $b_0 = b_8 = b_{16} = 1$ , others =0)); senior physicists will recall the three embeddings of  $SU(2)$  in flavour  $SU(3)$ , as Isospin, U-spin and V-spin. As eleven dimensional gravity is supposed to be the low energy limit of M-theory, we see that in this incompletely known theory octonions and a projective plane already play a role.

The mathematician Kostant [10] has proven that this is general phenomenon in coset spaces  $X = G/H$  where  $G$  is a semisimple Lie group and  $H$  reductive, with  $G$  and  $H$  of the same rank. Namely the identity representation of  $G$  induces  $\chi$  irreducible representations of  $H$ , where  $\chi$  is the Euler number of the coset space  $X$ ; this number is also the quotient of the orders of the Weyl reflection groups of  $G$  resp.  $H$ . These representations arrange themselves as the Spin representation of the  $SO(\dim X)$  group; here  $\dim X = 52 - 36 = 16$ , and the result of Kostant is, in our case

$$\begin{aligned} Spin_L(16) - Spin_R(16) &= h + C - \psi \\ 128 - 128 &= 44 + 84 - 128 \end{aligned} \tag{31}$$

Many supermultiplets (not all) can be understood in this way, see also [11]. This links up supersymmetry, spinors and octonions in an intricate way, not well elucidated up to now.

As is it well known, the 11D Suga multiplet is too small to encompass the spectrum of the standard model. So the question arises whether there is a different coset space furnishing a more realistic supersymmetric multiplet.

We claim the former space (28) or more precisely

$$Y := \frac{E_6/Z_3}{Spin^c(10)} \tag{32}$$

(where  $Z_3$  is the center of  $E_6$  and  $Spin^c(n) := Spin(n) \times_{Z_2} U(1)$ ) is a better candidate, although it contains, on the face of it, too many states. The plane  $Y$  corresponds precisely to the complexification of the Moufang plane [12], and the Euler number is

$$\chi(Y) = \#Weyl \text{ group of } E_6 / \#Weyl \text{ group of } D_5 = 51840/1920 = 27 \tag{33}$$

Therefore the Id irrep of  $E_6$  generates, in a supersymmetric fashion, 27  $2^{16}$ -dim irreps of  $SO(10) \times U(1)$ ! They are obtained by the skew products of the Spin irrep of  $SO(10)$ , of complex dimension 16 (with an irrelevant “charge” label associated to the  $U(1)$  subgroup). The total splitting had been calculated by I. Bars [13] in another context, and we just write it for the record:



$$(1 - 1)^{16} = 1 - 16 + \binom{16}{2} - \binom{16}{3} \dots \pm \dots - \binom{16}{15} + 1 = 32768 - 32768 \quad (34)$$

corresponding to the p-forms in  $C^{16}$  (the  $\text{Spin}(10)$  irrep is complex). The split with respect to  $O(10) \times O(2)$  is

SU(16)	Spin(10)	O(9)	O(8)	F-Th particle	
1	1	1	1	scalar	
-16	16	16	$8_L + 8_R$	spinor	
+120	120	84+36	$56+28, 28 + 8 + \dots$	3-form	
-560	560	432 + 128	...	hypergravitino	
1820	770+ 1050	924 + ...	...	Weyl Tensor	(35)
				+ self-6-form	
-4368	3696 + 672	2560+ ... +672			
8008	4312+3696	2457+...			
-11440	8800+2640	5040+...			
+12870	4125+8085	3900 +...			
	+ 660				

plus the conjugate irreps for the  $k = 9$  to 16 skew tensors of  $\text{SU}(16)$ .

There is no simple way to relate these representations to known particles; there must be a particular truncation, different from the naive square root (which will reproduce conventional 11D SUGRA) from  $2^{16}$  to  $256 = 2^8$  states, as required by the minimal supersymmetric model. We only remark here that among the spectrum of  $O(10) \times U(1)$  multiplet neither the graviton (55-dim) nor the gravitino (144-dim) appear, in agreement (if vaguely) with the idea that the supersymmetry is realized first without gravitation; in this sense  $F$ -theory (in 12 dimensions) fares better than  $M$  (in 11). There is also the feature that  $E_6$  plays a role, and it is the maximal group well fitted for Grand Unified Theories, GUTs. And, for numerologists,  $\dim \text{Poincaré} \sim \text{de Sitter / AdS in } (2, 10) \text{ space} = \dim E_6 = 78$ .

Finally, there is another intriguing feature, related to the mentioned square root: The SUGRA 11D multiplet can be seen as the "square" of the Yang-Mills multiplet in ten dimensions (8 effective)

$$(vector - spin_L) \times (vector - spin_R) = 44 + 1 + 28 + 8 + 56 - 8 - 8 - 56 - 56$$

that is, the graviton+dilaton+vector+2-form+3-form Bose content of IIA SUGRA in 10D +plus fermions (two spinors (8) plus two gravitinos (56)). Now this "oxidises" to 11d N=1 SUGRA as known:

graviton + dilaton + vector in 10D = graviton in 11D (44=35+8+1)

2 - & 3 - form in 10D = 3 - form in 11D (84 = 56 + 28)

*L & R spinor + gravitinos in 10D = graviton in 11D* (2·8 + 2·56 = 128)

Now Bars [13] has shown that another square produces the supermultiplet in (2, 10) dimensions!

In toto

$$[(vect - spin_L) \times (vect - spin_R)]^2 = Y - multiplets\ of\ O(10) \times O(2)$$

$$[(8 - 8) \times (8 - 8)]^2 = (16 \times 16)^2 = 32768 - 32768$$

So it seems that the natural extension of the fundamental supersymmetric 10D multiplet, given by the triality of  $O(8)$ , gives in the fourth power the supermultiplet of the symmetric space  $Y$  in 12D, but now with two times; this also shows, *inter alia*, that IIB string theory fits naturally in F-theory.

We leave this at that; we have not succeeded in truncating the enormous multiplet in a realistic way; but the issue is worth pursuing...

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